

# ON CLUSTER VARIABLES OF RANK TWO ACYCLIC CLUSTER ALGEBRAS

KYUNGYONG LEE

**ABSTRACT.** In this note, we find an explicit formula for the Laurent expression of cluster variables of coefficient-free rank two cluster algebras associated with the matrix  $\begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}$ , and show that a large number of coefficients are non-negative. As a corollary, we obtain an explicit expression for the Euler-Poincaré characteristics of the corresponding quiver Grassmannians.

**2010 Mathematics Subject Classification :** 13F60, 16G20.

**Keywords :** Cluster algebras, quiver representations, generalized Kronecker quiver.

## 1. INTRODUCTION

Let  $b, c$  be positive integers and  $x_1, x_2$  be indeterminates. The (coefficient-free) *cluster algebra*  $\mathcal{A}(b, c)$  is the subring of the field  $\mathbb{Q}(x_1, x_2)$  generated by the elements  $x_m$ ,  $m \in \mathbb{Z}$  satisfying the recurrence relations:

$$x_{n+1} = \begin{cases} (x_n^b + 1)/x_{n-1} & \text{if } n \text{ is odd,} \\ (x_n^c + 1)/x_{n-1} & \text{if } n \text{ is even.} \end{cases}$$

The elements  $x_m$ ,  $m \in \mathbb{Z}$  are called the cluster variables of  $\mathcal{A}(b, c)$ . Fomin and Zelevinsky [3] introduced cluster algebras and proved the Laurent phenomenon whose special case says that for every  $m \in \mathbb{Z}$  the cluster variable  $x_m$  can be expressed as a Laurent polynomial of  $x_1^{\pm 1}$  and  $x_2^{\pm 1}$ . In addition, they conjectured that the coefficients of monomials in the Laurent expression of  $x_m$  are non-negative integers. When  $bc \leq 4$ , Sherman-Zelevinsky [7] and independently Musiker-Propp [5] proved the conjecture. Moreover in this case the explicit combinatorial formulas for the coefficients are known. In this paper, we find an explicit formula for the coefficients when  $b = c \geq 2$ , and show that a large number of coefficients are non-negative.

As we will frequently use product forms, we say a few words about our convention. When we have any integer  $A$  and any function  $f(i)$  of  $i$ , the product  $\prod_{i=A}^{A-1} f(i)$  will be defined to be 1.

Before we state our main results, we need some definitions.

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Research partially supported by NSF grant DMS 0901367.

**Definition 1.** For arbitrary (possibly negative) integers  $A, B$ , we define the modified binomial coefficient as follows.

$$\left[ \begin{matrix} A \\ B \end{matrix} \right] := \begin{cases} \prod_{i=0}^{A-B-1} \frac{A-i}{A-B-i}, & \text{if } A > B \\ 1, & \text{if } A = B \\ 0, & \text{if } A < B. \end{cases}$$

□

If  $A \geq 0$  then  $\left[ \begin{matrix} A \\ B \end{matrix} \right] = \left[ \begin{matrix} A \\ A-B \end{matrix} \right]$  is just the usual binomial coefficient. In general,  $\left[ \begin{matrix} A \\ A-B \end{matrix} \right]$  is equal to the generalized binomial coefficient  $\binom{A}{B}$ . But in this paper we use our modified binomial coefficients to avoid too complicated expressions.

**Definition 2.** Let  $\{a_n\}$  be the sequence defined by the recurrence relation

$$a_n = ca_{n-1} - a_{n-2},$$

with the initial condition  $a_1 = 0, a_2 = 1$ . If  $c = 2$  then  $a_n = n - 1$ . When  $c > 2$ , it is easy to see that

$$a_n = \frac{1}{\sqrt{c^2 - 4}} \left( \frac{c + \sqrt{c^2 - 4}}{2} \right)^{n-1} - \frac{1}{\sqrt{c^2 - 4}} \left( \frac{c - \sqrt{c^2 - 4}}{2} \right)^{n-1} = \sum_{i \geq 0} (-1)^i \binom{n-2-i}{i} c^{n-2-2i}.$$

□

**Remark 3.** It is easy to show that for any  $n$ ,

$$(1.1) \quad a_{n-1}a_{n-3} - a_{n-2}^2 = -1,$$

which we will use later.

Our main result is the following.

**Theorem 4.** Assume that  $b = c \geq 2$ . Let  $n \geq 3$ . Then

$$(1.2) \quad x_n = x_1^{-a_{n-1}} x_2^{-a_{n-2}} \sum_{e_1, e_2} \sum_{t_0, t_1, \dots, t_{n-4}} \left[ \left( \prod_{i=0}^{n-4} \left[ \begin{matrix} a_{i+1} - cs_i \\ t_i \end{matrix} \right] \right) \right. \\ \left. \times \left[ \begin{matrix} a_{n-2} - cs_{n-3} \\ a_{n-2} - cs_{n-3} - e_2 + s_{n-4} \end{matrix} \right] \left[ \begin{matrix} -a_{n-3} + ce_2 \\ -a_{n-3} + ce_2 - e_1 + s_{n-3} \end{matrix} \right] x_1^{c(a_{n-2}-e_2)} x_2^{ce_1} \right],$$

where

$$s_i = \sum_{j=0}^{i-1} a_{i-j+1} t_j,$$

and the summations run over all integers  $e_1, e_2, t_0, \dots, t_{n-4}$  satisfying

$$(1.3) \quad \begin{cases} 0 \leq t_i \leq a_{i+1} - cs_i \ (0 \leq i \leq n-4), \\ 0 \leq a_{n-2} - cs_{n-3} - e_2 + s_{n-4} \leq a_{n-2} - cs_{n-3}, \text{ and} \\ e_2 a_{n-1} - e_1 a_{n-2} \geq 0. \end{cases}$$

Since  $\begin{bmatrix} A \\ B \end{bmatrix} \neq 0$  if and only if  $A \geq B$ , we may add the condition  $0 \geq -e_1 + s_{n-3}$  to (1.3). Then the summation in the statement is guaranteed to be a finite sum. A referee remarks that  $F$ -polynomials have similar expressions. As he pointed out, the expression without (1.3) is an easy consequence of the formula (6.28) in the paper [4] by Fomin and Zelevinsky, and the one with  $e_2 a_{n-1} - e_1 a_{n-2} \geq 0$  is a consequence of [7, Proposition 3.5] in the paper by Sherman and Zelevinsky. Our contribution is to show that all the modified binomial coefficients in (1.2) except for the last one are non-negative.

As a corollary to Theorem 4, we obtain an expression for the Euler-Poincaré characteristic of the variety  $\text{Gr}_{(e_1, e_2)}(M(n))$  of all subrepresentations of dimension  $(e_1, e_2)$  in a unique (up to an isomorphism) indecomposable  $Q_c$ -representation  $M(n)$  of dimension  $(a_{n-1}, a_{n-2})$ , where  $Q_c$  is the generalized Kronecker quiver with two vertices 1 and 2, and  $c$  arrows from 1 to 2. We use a result of Caldero and Zelevinsky [2, Theorem 3.2 and (3.5)].

**Theorem 5** (Caldero and Zelevinsky). *The cluster variable  $x_n$  is equal to*

$$x_1^{-a_{n-1}} x_2^{-a_{n-2}} \sum_{e_1, e_2} \chi(\text{Gr}_{(e_1, e_2)}(M(n))) x_1^{c(a_{n-2} - e_2)} x_2^{ce_1}.$$

**Corollary 6.** *Assume that  $b = c \geq 2$ . For any  $(e_1, e_2)$  and  $n \geq 3$ , the Euler-Poincaré characteristic of  $\text{Gr}_{(e_1, e_2)}(M(n))$  is equal to*

$$(1.4) \quad \sum_{t_0, t_1, \dots, t_{n-4}} \left[ \left( \prod_{i=0}^{n-4} \begin{bmatrix} a_{i+1} - cs_i \\ t_i \end{bmatrix} \right) \begin{bmatrix} a_{n-2} - cs_{n-3} \\ a_{n-2} - cs_{n-3} - e_2 + s_{n-4} \end{bmatrix} \begin{bmatrix} -a_{n-3} + ce_2 \\ -a_{n-3} + ce_2 - e_1 + s_{n-3} \end{bmatrix} \right],$$

where the summation runs over all integers  $t_0, \dots, t_{n-4}$  satisfying

$$(1.5) \quad \begin{cases} 0 \leq t_i \leq a_{i+1} - cs_i \ (0 \leq i \leq n-4), \text{ and} \\ 0 \leq a_{n-2} - cs_{n-3} - e_2 + s_{n-4} \leq a_{n-2} - cs_{n-3}. \end{cases}$$

**Corollary 7.** *Assume that  $b = c \geq 3$ . Let  $n \geq 3$ . For any  $(e_1, e_2)$  with  $e_2 \geq \frac{a_{n-3}}{c}$ , the Euler-Poincaré characteristic of  $\text{Gr}_{(e_1, e_2)}(M(n))$  is non-negative.*

*Acknowledgement.* We are grateful to Grégoire Dupont for valuable discussions and correspondence. We also thank anonymous referees for their useful suggestions and helpful comments.

## 2. PROOFS

We actually prove the following statement, which is equivalent to Theorem 4 but simpler to prove.

**Theorem 8.** *Assume that  $b = c \geq 2$ . Let  $n \geq 3$ . Then*

$$(2.1) \quad x_n = x_1^{-a_{n-1}} x_2^{-a_{n-2}} \sum_{t_0, t_1, \dots, t_{n-2}} \left[ \left( \prod_{i=0}^{n-2} \begin{bmatrix} a_{i+1} - cs_i \\ t_i \end{bmatrix} \right) x_1^{cs_{n-2}} x_2^{c(a_{n-1} - s_{n-1})} \right],$$

where

$$s_i = \sum_{j=0}^{i-1} a_{i-j+1} t_j,$$

and the summation runs over all integers  $t_0, \dots, t_{n-2}$  satisfying

$$(2.2) \quad \begin{cases} 0 \leq t_i \leq a_{i+1} - cs_i (0 \leq i \leq n-3), \text{ and} \\ s_{n-1} a_{n-2} - s_{n-2} a_{n-1} \geq 0. \end{cases}$$

**Lemma 9.** *Theorem 8 is equivalent to Theorem 4.*

*Proof.* In (1.2), if we substitute  $a_{n-i} - s_{n-i}$  for  $e_i$  ( $i = 1, 2$ ), we obtain (2.1). Note that the coefficient of  $t_{n-i-1}$  in  $s_{n-i}$  is equal to 1. Hence, as  $e_i$  runs over integers, so does  $t_{n-i-1}$ .  $\square$

*Proof of Theorem 8.* It is not hard to check the statement for  $n = 3, 4, 5$ . When  $n \geq 5$ , we use induction on  $n$ .

Suppose that the statement holds for  $n$  or less. Then by the obvious shift, we have

$$x_{n+1} = x_2^{-a_{n-1}} x_3^{-a_{n-2}} \sum_{t_0, t_1, \dots, t_{n-2}} \left[ \left( \prod_{i=0}^{n-2} \begin{bmatrix} a_{i+1} - cs_i \\ t_i \end{bmatrix} \right) x_2^{cs_{n-2}} x_3^{c(a_{n-1} - s_{n-1})} \right],$$

where the summation runs over all integers  $t_0, \dots, t_{n-2}$  satisfying (2.2).

Substituting  $\frac{x_2^c + 1}{x_1}$  into  $x_3$ , we get

$$\begin{aligned} x_{n+1} &= x_2^{-a_{n-1}} \sum_{t_0, t_1, \dots, t_{n-2}} \left[ \left( \prod_{i=0}^{n-2} \begin{bmatrix} a_{i+1} - cs_i \\ t_i \end{bmatrix} \right) x_2^{cs_{n-2}} \left( \frac{x_2^c + 1}{x_1} \right)^{c(a_{n-1} - s_{n-1}) - a_{n-2}} \right], \\ &= x_2^{-a_{n-1}} \sum_{t_0, t_1, \dots, t_{n-2}} \left[ \left( \prod_{i=0}^{n-2} \begin{bmatrix} a_{i+1} - cs_i \\ t_i \end{bmatrix} \right) x_2^{cs_{n-2}} \left( \frac{x_2^c + 1}{x_1} \right)^{a_n - cs_{n-1}} \right], \\ &= x_2^{-a_{n-1}} \sum_{t_0, t_1, \dots, t_{n-2}} \left[ \left( \prod_{i=0}^{n-2} \begin{bmatrix} a_{i+1} - cs_i \\ t_i \end{bmatrix} \right) \sum_{t_{n-1} \in \mathbb{Z}} \begin{bmatrix} a_n - cs_{n-1} \\ t_{n-1} \end{bmatrix} (x_2^c)^{a_n - cs_{n-1} - t_{n-1}} x_1^{cs_{n-1} - a_n} x_2^{cs_{n-2}} \right], \\ &= x_2^{-a_{n-1}} \sum_{t_0, t_1, \dots, t_{n-1}} \left[ \left( \prod_{i=0}^{n-1} \begin{bmatrix} a_{i+1} - cs_i \\ t_i \end{bmatrix} \right) (x_2^c)^{a_n - cs_{n-1} - t_{n-1}} x_1^{cs_{n-1} - a_n} x_2^{cs_{n-2}} \right], \\ &= x_1^{-a_n} x_2^{-a_{n-1}} \sum_{t_0, t_1, \dots, t_{n-1}} \left[ \left( \prod_{i=0}^{n-1} \begin{bmatrix} a_{i+1} - cs_i \\ t_i \end{bmatrix} \right) x_1^{cs_{n-1}} x_2^{c(a_n - s_n)} \right], \end{aligned}$$

where the last equality follows from

$$s_n = \sum_{j=1}^{n-1} a_{n-j+1} t_j = \sum_{j=1}^{n-1} (ca_{n-j} - a_{n-j-1}) t_j = cs_{n-1} - s_{n-2} + t_{n-1}.$$

If one worries about convergence of the sum, then we could have begun by assuming that  $|x_2| < 1$ , but since we will eventually show that the sum is finite, the convergence should not be a problem.

Remember that  $t_0, \dots, t_{n-2}$  satisfy (2.2). By identifying  $a_{n+1-i} - s_{n+1-i}$  with  $e_i$  ( $i = 1, 2$ ), Proposition 10 implies that even if  $t_{n-2}$  and  $t_{n-1}$  run over the only integers satisfying  $s_n a_{n-1} - s_{n-1} a_n \geq 0$ , we get the same result. On the other hand, in order to prove that Theorem 8 holds for  $n + 1$ , we need to show that  $t_{n-2}$  is enough to run over  $0 \leq t_{n-2} \leq a_{n-1} - cs_{n-2}$ . The second inequality is clear, because otherwise  $\begin{bmatrix} a_{n-1} - cs_{n-2} \\ t_{n-2} \end{bmatrix} = 0$  by Definition 1. So we want to show that

$$(2.3) \quad \sum_{t_0, t_1, \dots, t_{n-1}} \left[ \left( \prod_{i=0}^{n-1} \begin{bmatrix} a_{i+1} - cs_i \\ t_i \end{bmatrix} \right) x_1^{cs_{n-1}} x_2^{c(a_n - s_n)} \right] = 0,$$

where the summation runs over all integers  $t_0, \dots, t_{n-1}$  satisfying

$$(2.4) \quad \begin{cases} 0 \leq t_i \leq a_{i+1} - cs_i \ (0 \leq i \leq n-3), \\ s_{n-1} a_{n-2} - s_{n-2} a_{n-1} \geq 0, \\ t_{n-2} \leq a_{n-1} - cs_{n-2} < 0, \text{ and} \\ s_n a_{n-1} - s_{n-1} a_n \geq 0. \end{cases}$$

To do this, we will show that  $a_n - cs_{n-1} < 0$ . Suppose to the contrary that  $a_n - cs_{n-1} \geq 0$ . First of all, we have

$$(2.5) \quad \begin{aligned} & a_{n-3} s_{n-2} - a_{n-2} (a_{n-1} - s_{n-1}) \\ &= ca_{n-2} s_{n-2} - a_{n-1} s_{n-2} - a_{n-2} (a_{n-1} - s_{n-1}) \\ &> a_{n-2} a_{n-1} - a_{n-1} s_{n-2} - a_{n-2} (a_{n-1} - s_{n-1}) \quad \text{since } a_{n-1} - cs_{n-2} < 0 \\ &= s_{n-1} a_{n-2} - s_{n-2} a_{n-1} \underset{\text{by (2.4)}}{\geq} 0. \end{aligned}$$

Then

$$(2.6) \quad \begin{aligned} & a_{n-2} s_{n-1} - a_{n-1} s_{n-2} \\ & \underset{\text{by (2.5)}}{<} a_{n-2} s_{n-1} - a_{n-1} \frac{a_{n-2}}{a_{n-3}} (a_{n-1} - s_{n-1}) \\ &= a_{n-2} \left( a_{n-1} - \left( 1 + \frac{a_{n-1}}{a_{n-3}} \right) (a_{n-1} - s_{n-1}) \right) \\ &= a_{n-2} \left( a_{n-1} - \left( 1 + \frac{a_{n-1}}{a_{n-3}} \right) \frac{a_{n-2} + a_n - cs_{n-1}}{c} \right) \\ &\leq a_{n-2} \left( a_{n-1} - \frac{a_{n-3} + a_{n-1}}{a_{n-3}} \frac{a_{n-2}}{c} \right) \quad \text{since } a_n - cs_{n-1} \geq 0 \\ &= a_{n-2} \left( a_{n-1} - \frac{a_{n-2}^2}{a_{n-3}} \right) \\ &\underset{\text{by (1.1)}}{=} -\frac{a_{n-2}}{a_{n-3}} < 0, \end{aligned}$$

which contradicts  $s_{n-1} a_{n-2} - s_{n-2} a_{n-1} \geq 0$ . Hence

$$(2.7) \quad a_n - cs_{n-1} < 0.$$

Next we show that  $s_{n-2} > a_n - s_n$ . Suppose to the contrary that  $s_{n-2} \leq a_n - s_n$ . Then

$$\begin{aligned} a_{n-1} - cs_{n-2} &\geq a_{n-1} - c(a_n - s_n) \stackrel{\text{by (2.4)}}{\geq} a_{n-1} - c \frac{(a_{n-1} - s_{n-1})a_n}{a_{n-1}} \\ &\stackrel{\text{by (1.1)}}{=} \frac{a_n a_{n-2} + 1}{a_{n-1}} - c \frac{(a_{n-1} - s_{n-1})a_n}{a_{n-1}} = \frac{a_n}{a_{n-1}} (a_{n-2} - c(a_{n-1} - s_{n-1})) + \frac{1}{a_{n-1}} \\ &= \frac{a_n}{a_{n-1}} (cs_{n-1} - a_n) + \frac{1}{a_{n-1}} \stackrel{\text{by (2.7)}}{>} 0, \end{aligned}$$

which contradicts  $a_{n-1} - cs_{n-2} < 0$  in (2.4). Thus  $s_{n-2} > a_n - s_n$ , so we have

$$a_n - cs_{n-1} < s_n + s_{n-2} - cs_{n-1} = t_{n-1},$$

which gives  $\begin{bmatrix} a_n - cs_{n-1} \\ t_{n-1} \end{bmatrix} = 0$ . Therefore,  $(2.3) = 0$ .

So far we have proved that

$$x_{n+1} = x_1^{-a_n} x_2^{-a_{n-1}} \sum_{t_0, t_1, \dots, t_{n-1}} \left[ \left( \prod_{i=0}^{n-1} \begin{bmatrix} a_{i+1} - cs_i \\ t_i \end{bmatrix} \right) x_1^{cs_{n-1}} x_2^{c(a_n - s_n)} \right],$$

where the summation runs over all integers  $t_0, \dots, t_{n-1}$  satisfying

$$(2.8) \quad \begin{cases} 0 \leq t_i \leq a_{i+1} - cs_i \ (0 \leq i \leq n-2), \\ s_{n-1}a_{n-2} - s_{n-2}a_{n-1} \geq 0, \text{ and} \\ s_n a_{n-1} - s_{n-1}a_n \geq 0. \end{cases}$$

But we do not have to include  $s_{n-1}a_{n-2} - s_{n-2}a_{n-1} \geq 0$  in (2.8), because  $0 \leq t_i \leq a_{i+1} - cs_i$  ( $0 \leq i \leq n-2$ ) imply  $s_{n-1}a_{n-2} - s_{n-2}a_{n-1} \geq 0$  as follows.

$$\begin{aligned} s_{n-1}a_{n-2} - s_{n-2}a_{n-1} &= (cs_{n-2} - s_{n-3} + t_{n-2})a_{n-2} - s_{n-2}a_{n-1} \\ &= (s_{n-2}a_{n-3} - s_{n-3}a_{n-2}) + t_{n-2}a_{n-2} = \dots = (s_2a_1 - s_1a_2) + \sum_{i=2}^{n-2} t_i a_i = 0 + \sum_{i=2}^{n-2} t_i a_i \geq 0. \end{aligned}$$

This completes the proof modulo Proposition 10.  $\square$

**Proposition 10.** Fix four integers  $c(\geq 1), n(\geq 3), e_1$  and  $e_2$  satisfying  $e_2 a_{n-1} - e_1 a_{n-2} < 0$ . Then

$$(2.9) \quad \sum_{t_0, t_1, \dots, t_{n-4}} \left[ \left( \prod_{i=0}^{n-4} \begin{bmatrix} a_{i+1} - cs_i \\ t_i \end{bmatrix} \right) \times \begin{bmatrix} a_{n-2} - cs_{n-3} \\ a_{n-2} - cs_{n-3} - e_2 + s_{n-4} \end{bmatrix} \begin{bmatrix} -a_{n-3} + ce_2 \\ -a_{n-3} + ce_2 - e_1 + s_{n-3} \end{bmatrix} \right] = 0,$$

where the summation runs over all integers  $t_0, \dots, t_{n-4}$  satisfying

$$0 \leq t_i \leq a_{i+1} - cs_i \ (0 \leq i \leq n-4).$$

This is a consequence of [7, Proposition 3.5] in the paper by Sherman and Zelevinsky. One also may give a geometric proof. Actually one can show that  $e_2 a_{n-1} - e_1 a_{n-2} < 0$  implies

$\langle (e_1, e_2), (a_{n-1} - e_1, a_{n-2} - e_2) \rangle < 0$ , where  $\langle \cdot, \cdot \rangle$  is the Euler inner product (for instance, see [6]). Then the assertion follows from a result of Schofield [6, Section 3], which says that

$$\dim \text{Gr}_{(e_1, e_2)} M(n) = \langle (e_1, e_2), (a_{n-1} - e_1, a_{n-2} - e_2) \rangle.$$

Hence if  $\langle (e_1, e_2), (a_{n-1} - e_1, a_{n-2} - e_2) \rangle < 0$  then  $\text{Gr}_{(e_1, e_2)} M(n)$  is empty, so its Euler characteristic  $\chi(\text{Gr}_{(e_1, e_2)} M(n))$  is obviously zero, which is equivalent to (2.9) = 0 by [2, Theorem 3.2 and (3.5)].

However we will give a different proof, because we want to keep the exposition self-contained. Before we give the proof, we need some lemmas.

**Lemma 11.** *Let  $A, B, q, m$  be (possibly negative) integers with  $A + B \geq q \geq 0$ . Let  $P(w) \in \mathbb{Q}[w]$  be any polynomial of  $w$  of degree  $q$ . Then*

$$\sum_{w \in \mathbb{Z}} P(w) \begin{bmatrix} A \\ w \end{bmatrix} \begin{bmatrix} B \\ m - w \end{bmatrix} = \sum_{w \in \mathbb{Z}} P(w) \begin{bmatrix} A \\ A - w \end{bmatrix} \begin{bmatrix} B \\ B - m + w \end{bmatrix}.$$

*Proof.* Since any polynomial of  $w$  of degree  $q$  is a  $\mathbb{Q}$ -linear combination of  $\prod_{i=0}^{p-1} (w - i)$  ( $0 \leq p \leq q$ ), it is enough to show that for any  $p$  ( $0 \leq p \leq q$ ), we have

$$\sum_{w \in \mathbb{Z}} \left( \prod_{i=0}^{p-1} (w - i) \right) \begin{bmatrix} A \\ w \end{bmatrix} \begin{bmatrix} B \\ m - w \end{bmatrix} = \sum_{w \in \mathbb{Z}} \left( \prod_{i=0}^{p-1} (w - i) \right) \begin{bmatrix} A \\ A - w \end{bmatrix} \begin{bmatrix} B \\ B - m + w \end{bmatrix}.$$

If  $A, B \geq 0$  then the equality is trivial. So we assume that either  $A < 0$  or  $B < 0$ . Without loss of generality, we assume  $B < 0$ . Since  $A + B \geq q$ , we have  $A \geq q$  hence  $A \geq p$ . Then

$$\begin{aligned} & \sum_{w \in \mathbb{Z}} \prod_{i=0}^{p-1} (w - i) \begin{bmatrix} A \\ w \end{bmatrix} \begin{bmatrix} B \\ m - w \end{bmatrix} = \sum_{w \in \mathbb{Z}} \prod_{i=0}^{p-1} (w - i) \begin{bmatrix} A \\ A - w \end{bmatrix} \begin{bmatrix} B \\ m - w \end{bmatrix} \quad (\text{since } A \geq 0) \\ &= \sum_{w \in \mathbb{Z}} \prod_{i=0}^{p-1} (w - i) \prod_{i=0}^{w-1} \frac{A - i}{w - i} \begin{bmatrix} B \\ m - w \end{bmatrix} = \sum_{w \in \mathbb{Z}} \prod_{i=0}^{p-1} (A - i) \prod_{i=p}^{w-1} \frac{A - i}{w - i} \begin{bmatrix} B \\ m - w \end{bmatrix} \\ &= \sum_{w \in \mathbb{Z}} \prod_{i=0}^{p-1} (A - i) \begin{bmatrix} A - p \\ w - p \end{bmatrix} \begin{bmatrix} B \\ m - w \end{bmatrix} = \prod_{i=0}^{p-1} (A - i) \begin{bmatrix} A + B - p \\ m - p \end{bmatrix} \\ &= \prod_{i=0}^{p-1} (A - i) \begin{bmatrix} A + B - p \\ A + B - m \end{bmatrix} \quad (\text{since } A + B - p \geq 0) \\ &= \sum_{w \in \mathbb{Z}} \prod_{i=0}^{p-1} (A - i) \begin{bmatrix} A - p \\ A - w \end{bmatrix} \begin{bmatrix} B \\ B - m + w \end{bmatrix} \\ &= \sum_{w \in \mathbb{Z}} \prod_{i=0}^{p-1} (A - i) \begin{bmatrix} A - p \\ w - p \end{bmatrix} \begin{bmatrix} B \\ B - m + w \end{bmatrix} \quad (\text{since } A - p \geq 0) \\ &= \sum_{w \in \mathbb{Z}} \prod_{i=0}^{p-1} (w - i) \begin{bmatrix} A \\ A - w \end{bmatrix} \begin{bmatrix} B \\ B - m + w \end{bmatrix}. \end{aligned}$$

□

**Lemma 12.** Fix four integers  $c(\geq 1), n(\geq 3), e_1$  and  $e_2$ . Let  $w_{n-2} = 0$ . For any  $-1 \leq j \leq n-4$ , define  $f(j)$  by

$$\begin{aligned} f(j) = & \sum_{t_0, t_1, \dots, t_j} \sum_{w_1, \dots, w_{n-j-4}} \left[ \left( \prod_{i=0}^j \begin{bmatrix} a_{i+1} - cs_i \\ t_i \end{bmatrix} \right) \right. \\ & \times \begin{bmatrix} a_{j+2} - cs_{j+1} \\ a_{j+2} - cs_{j+1} + s_j - (e_2 a_{n-2-j} - e_1 a_{n-3-j} - v_{n-3-j}) \end{bmatrix} \\ & \times \begin{bmatrix} -a_{j+1} + c(e_2 a_{n-2-j} - e_1 a_{n-3-j} - v_{n-3-j}) \\ s_{j+1} - a_{j+1} + e_2 a_{n-1-j} - e_1 a_{n-2-j} - v_{n-2-j} + w_{n-3-j} \end{bmatrix} \\ & \times \prod_{i=j+2}^{n-3} \begin{bmatrix} -a_i + c(e_2 a_{n-i-1} - e_1 a_{n-i-2} - v_{n-i-2}) \\ -a_i + c(e_2 a_{n-i-1} - e_1 a_{n-i-2} - v_{n-i-2}) - w_{n-i-2} \end{bmatrix} \left. \right], \end{aligned}$$

where

$$v_i = \sum_{j=1}^{i-1} a_{i-j+1} w_j,$$

and the summations run over all integers  $t_0, \dots, t_j, w_1, \dots, w_{n-j-4}$  satisfying

$$a_{i+1} - cs_i \geq 0 \quad (0 \leq i \leq j) \quad \text{and} \quad w_i \geq 0 \quad (1 \leq i \leq n-j-4).$$

Then  $f(-1) = f(0) = \dots = f(n-4)$ .

*Proof.* This is essentially a change of variables, together with the help of Lemma 11. We frequently use

$$\begin{aligned} (2.10) \quad & a_1 = 0, \quad a_2 = 1, \quad a_i = ca_{i-1} - a_{i-2}, \\ & s_i = \sum_{j=1}^{i-1} a_{i-j+1} t_j = \sum_{j=1}^{i-1} (ca_{i-j} - a_{i-j-1}) t_j = cs_{i-1} - s_{i-2} + t_{i-1}, \quad \text{and} \\ & v_i = \sum_{j=1}^{i-1} a_{i-j+1} w_j = \sum_{j=1}^{i-1} (ca_{i-j} - a_{i-j-1}) w_j = cv_{i-1} - v_{i-2} + w_{i-1}. \end{aligned}$$

We will give a detailed proof for  $f(n-4) = f(n-5)$ . The rest of the equalities can be obtained similarly.

$$\begin{aligned} f(n-4) = & \sum_{t_0, t_1, \dots, t_{n-5}} \left[ \left( \prod_{i=0}^{n-5} \begin{bmatrix} a_{i+1} - cs_i \\ t_i \end{bmatrix} \right) \right. \\ & \times \sum_{t_{n-4} \in \mathbb{Z}} \begin{bmatrix} a_{n-3} - cs_{n-4} \\ t_{n-4} \end{bmatrix} \begin{bmatrix} a_{n-2} - cs_{n-3} \\ a_{n-2} - cs_{n-3} - e_2 + s_{n-4} \end{bmatrix} \begin{bmatrix} -a_{n-3} + ce_2 \\ -a_{n-3} + ce_2 - e_1 + s_{n-3} \end{bmatrix} \left. \right]. \end{aligned}$$

Since  $a_{n-3} - cs_{n-4} \geq 0$ , we have

$$\begin{aligned} f(n-4) = & \sum_{t_0, t_1, \dots, t_{n-5}} \left[ \left( \prod_{i=0}^{n-5} \begin{bmatrix} a_{i+1} - cs_i \\ t_i \end{bmatrix} \right) \right. \\ & \times \sum_{t_{n-4} \in \mathbb{Z}} \begin{bmatrix} a_{n-3} - cs_{n-4} \\ a_{n-3} - cs_{n-4} - t_{n-4} \end{bmatrix} \begin{bmatrix} a_{n-2} - cs_{n-3} \\ a_{n-2} - cs_{n-3} - e_2 + s_{n-4} \end{bmatrix} \begin{bmatrix} -a_{n-3} + ce_2 \\ -a_{n-3} + ce_2 - e_1 + s_{n-3} \end{bmatrix} \left. \right]. \end{aligned}$$



Substituting  $a_{n-3} - cs_{n-4} + s_{n-5} - (ce_2 - e_1) + w_1$  into  $t_{n-4}$ , we get

$$f(n-4) = \sum_{t_0, t_1, \dots, t_{n-5}} \left[ \left( \prod_{i=0}^{n-5} \begin{bmatrix} a_{i+1} - cs_i \\ t_i \end{bmatrix} \right) \right. \\ \left. \times \sum_{w_1 \in \mathbb{Z}} \begin{bmatrix} a_{n-3} - cs_{n-4} \\ -s_{n-5} + ce_2 - e_1 - w_1 \end{bmatrix} \begin{bmatrix} -a_{n-4} + c(ce_2 - e_1) - cw_1 \\ -a_{n-4} + c(ce_2 - e_1) - cw_1 - e_2 + s_{n-4} \end{bmatrix} \begin{bmatrix} -a_{n-3} + ce_2 \\ w_1 \end{bmatrix} \right].$$

Here  $\begin{bmatrix} -a_{n-4} + c(ce_2 - e_1) - cw_1 \\ -a_{n-4} + c(ce_2 - e_1) - cw_1 - e_2 + s_{n-4} \end{bmatrix}$  (if nonzero) can be regarded as a polynomial of  $w_1$  of degree  $e_2 - s_{n-4} \geq 0$ . Since

$$0 \leq e_2 - s_{n-4} \leq (a_{n-3} - cs_{n-4}) + (-a_{n-3} + ce_2),$$

we can apply Lemma 11. Then we obtain

$$\begin{aligned} f(n-4) &= \sum_{t_0, t_1, \dots, t_{n-5}} \left[ \left( \prod_{i=0}^{n-5} \begin{bmatrix} a_{i+1} - cs_i \\ t_i \end{bmatrix} \right) \right. \\ &\quad \times \sum_{w_1 \in \mathbb{Z}} \begin{bmatrix} a_{n-3} - cs_{n-4} \\ a_{n-3} - cs_{n-4} + s_{n-5} - (ce_2 - e_1) + w_1 \end{bmatrix} \begin{bmatrix} -a_{n-4} + c(ce_2 - e_1) - cw_1 \\ -a_{n-4} + c(ce_2 - e_1) - cw_1 - e_2 + s_{n-4} \end{bmatrix} \begin{bmatrix} -a_{n-3} + ce_2 \\ -a_{n-3} + ce_2 - w_1 \end{bmatrix} \Big] \\ &= \sum_{t_0, t_1, \dots, t_{n-5}} \left[ \left( \prod_{i=0}^{n-5} \begin{bmatrix} a_{i+1} - cs_i \\ t_i \end{bmatrix} \right) \right. \\ &\quad \times \sum_{w_1 \in \mathbb{Z}} \begin{bmatrix} a_{n-3} - cs_{n-4} \\ a_{n-3} - cs_{n-4} + s_{n-5} - (e_2a_3 - e_1a_2 - v_2) \end{bmatrix} \begin{bmatrix} -a_{n-4} + c(e_2a_3 - e_1a_2 - v_2) \\ s_{n-4} - a_{n-4} + e_2a_4 - e_1a_3 - v_3 + w_2 \end{bmatrix} \\ &\quad \times \begin{bmatrix} -a_{n-3} + c(e_2a_2 - e_1a_1 - v_1) \\ -a_{n-3} + c(e_2a_2 - e_1a_1 - v_1) - w_1 \end{bmatrix} \Big], \end{aligned}$$

where we have used (2.10). Since  $\begin{bmatrix} -a_{n-3} + c(e_2a_2 - e_1a_1 - v_1) \\ -a_{n-3} + c(e_2a_2 - e_1a_1 - v_1) - w_1 \end{bmatrix} = 0$  for  $w_1 < 0$ , we actually have

$$\begin{aligned} f(n-4) &= \sum_{t_0, t_1, \dots, t_{n-5}} \left[ \left( \prod_{i=0}^{n-5} \begin{bmatrix} a_{i+1} - cs_i \\ t_i \end{bmatrix} \right) \right. \\ &\quad \times \sum_{w_1 \geq 0} \begin{bmatrix} a_{n-3} - cs_{n-4} \\ a_{n-3} - cs_{n-4} + s_{n-5} - (e_2a_3 - e_1a_2 - v_2) \end{bmatrix} \begin{bmatrix} -a_{n-4} + c(e_2a_3 - e_1a_2 - v_2) \\ s_{n-4} - a_{n-4} + e_2a_4 - e_1a_3 - v_3 + w_2 \end{bmatrix} \\ &\quad \times \begin{bmatrix} -a_{n-3} + c(e_2a_2 - e_1a_1 - v_1) \\ -a_{n-3} + c(e_2a_2 - e_1a_1 - v_1) - w_1 \end{bmatrix} \Big], \end{aligned}$$

which is equal to  $f(n-5)$ .

Again since  $a_{n-4} - cs_{n-5} \geq 0$ , we have

$$\begin{aligned} f(n-5) &= \sum_{t_0, t_1, \dots, t_{n-6}} \sum_{w_1 \geq 0} \left[ \left( \prod_{i=0}^{n-6} \begin{bmatrix} a_{i+1} - cs_i \\ t_i \end{bmatrix} \right) \right. \\ &\quad \times \begin{bmatrix} a_{n-4} - cs_{n-5} \\ a_{n-4} - cs_{n-5} - t_{n-5} \end{bmatrix} \begin{bmatrix} a_{n-3} - cs_{n-4} \\ a_{n-3} - cs_{n-4} + s_{n-5} - (e_2a_3 - e_1a_2 - v_2) \end{bmatrix} \begin{bmatrix} -a_{n-4} + c(e_2a_3 - e_1a_2 - v_2) \\ s_{n-4} - a_{n-4} + e_2a_4 - e_1a_3 - v_3 + w_2 \end{bmatrix} \\ &\quad \times \begin{bmatrix} -a_{n-3} + c(e_2a_2 - e_1a_1 - v_1) \\ -a_{n-3} + c(e_2a_2 - e_1a_1 - v_1) - w_1 \end{bmatrix} \Big]. \end{aligned}$$

In the same manner as above, i.e. by substituting  $a_{n-4} - cs_{n-5} + s_{n-6} - (e_2a_4 - e_1a_3) + v_3$  into  $t_{n-5}$  and then applying Lemma 11, it is not hard to show that

$$f(n-5) = f(n-6).$$

Repeating this process, we eventually obtain the desired equalities.  $\square$

*Proof of Proposition 10.* By Lemma 12, the left-hand side of (2.9), which is  $f(n-4)$ , is equal to  $f(-1)$ . Since  $a_1 = 0$ , the first modified binomial coefficient in  $f(-1)$  is equal to

$$\begin{bmatrix} 0 \\ -(e_2 a_{n-1} - e_1 a_{n-2} - \sum_{j=1}^{n-3} a_{n-1-j} w_j) \end{bmatrix}.$$

Here  $\sum_{j=1}^{n-3} a_{n-1-j} w_j \geq 0$  since  $w_j \geq 0$ . Therefore, if  $e_2 a_{n-1} - e_1 a_{n-2} < 0$  then

$$\begin{bmatrix} 0 \\ -(e_2 a_{n-1} - e_1 a_{n-2} - \sum_{j=1}^{n-3} a_{n-1-j} w_j) \end{bmatrix} = 0$$

for any  $w_j \geq 0$ , which gives  $f(-1) = 0$ . This completes the proof.  $\square$

*Proof of Corollary 6.* Corollary 6 is an immediate consequence of Theorem 4 thanks to a result of Caldero and Zelevinsky [2, Theorem 3.2 and (3.5)]. If  $e_2 a_{n-1} - e_1 a_{n-2} < 0$  then following from the discussion after Proposition 10, we have  $(1.4) = 0 = \chi(\text{Gr}_{(e_1, e_2)} M(n))$ .  $\square$

*Proof of Corollary 7.* By (1.5), all the modified binomial coefficients except for the last one in (1.4) are non-negative. If  $e_2 \geq \frac{a_{n-3}}{c}$  then the last one also becomes non-negative. Therefore, Corollary 6 implies that  $\chi(\text{Gr}_{(e_1, e_2)} M(n))$  is non-negative.  $\square$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CONNECTICUT, STORRS, CT 06269

*E-mail address:* kyung1@purdue.edu